Math 115A Worksheet
Thursday, Oct 5 (Week 1)

1. Rewrite each of the following statements in the form of an if-then statement.

   (a) The matrix $A$ is invertible only if $A$ is square.

   If $A$ is invertible then $A$ is square.

   (b) $n^2$ is an even integer, provided that $n$ is an even integer.

   If $n$ is an even integer then $n^2$ is an even integer.

   (c) All odd numbers are prime.

   If $n$ is an odd number then $n$ is prime.

   (d) The side lengths of a right triangle satisfy the equation $a^2 + b^2 = c^2$, where $a$ and $b$ are the lengths of the legs and $c$ is the length of the hypotenuse.

   If $a$ and $b$ are the lengths of the legs of a right triangle and $c$ is the length of the hypotenuse, then $a^2+b^2=c^2$.

   (e) Let $f(x) = 2^x$. Then $f(x) = 0$ has a solution for some real number $x$.

   If $f(x)=2^x$ then there is some $x \in \mathbb{R}$ so that $f(x)=0$.

2. For each of the following statements, find the converse, contrapositive, and negation.

   (a) All rational numbers are real numbers. $p \Rightarrow q$

   $q \Rightarrow p$ Converse: All real numbers are rational.

   not $q \Rightarrow$ not $p$ Contrapositive: If $x$ is not real then $x$ is not rational.

   not $p \Rightarrow$ not $q$ Negation: If $x$ is not rational then $x$ is not real.

(b) If $x^2$ is even, then $x$ is even.

Converse: If $x$ is even then $x^2$ is even.

Contrapositive: If $x$ is odd then $x^2$ is odd.

Negation: If $x^2$ is odd then $x$ is odd.

(c) A prime number is odd, only if it is greater than 2. If a prime is odd, it's greater than 2

Converse: If a prime number is greater than 2, it's odd.

Contrapositive: If a prime number is at most 2, it's even.

Negation: If a prime number is even, it's at most 2.

3. Show each of the following statements is false by finding a counterexample, i.e. for each problem, you should:

- Find the negation of the statement, which is in the form "There exists..."
- Find an appropriate example to show that this negation is true (the original statement is false)
- **Explain why your chosen example has the desired properties.** Don't forget to do this even if it seems obvious.

(a) All positive numbers are rational.

Negation: There exists a positive number which is irrational.

Example: $\sqrt{2}$ is positive and irrational

Proof: we pick the positive solution $\sqrt{2} > 0$ to $x^2 = 2$.
If $\sqrt{2}$ was rational, then $\sqrt{2} = \frac{p}{q}$ for coprime positive integers $p$ and $q$. Hence $2q^2 = p^2$ so $p$ is even. Let $p = 2p'$. Then $2q^2 = 4(p')^2$, so $q^2 = 2(p')^2$ and $q$ is even. But $p, q$ were assumed to be coprime, so we have reached a contradiction and $\sqrt{2}$ is irrational.

(b) All square matrices are invertible.

Negation: There is a square matrix which is not invertible.

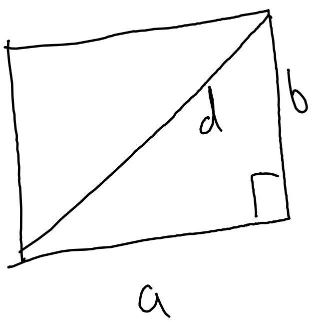
Example: $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Proof: $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is $2\times 2$, hence square. $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, so for no $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$!

Hence, $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ has no inverse.

4. Prove each of the claims below. Try to use only facts noted in the problem statement and basic algebra.

(a) Suppose that $R$ is a rectangle and $d$ is the length of the diagonal. Prove that if the area of $R$ is equal to $d^2/2$, then the rectangle is a square. (Use the Pythagorean theorem)




Let $a$ and $b$ be the side lengths of $R$.
By the Pythagorean Theorem, $d^2 = a^2 + b^2$.
Furthermore, the area $A$ of $R$ is $A = ab$.
Suppose $A = d^2/2$. Then $ab = \frac{a^2+b^2}{2}$, so $a^2+b^2 = 2ab$ and $a^2 - 2ab + b^2 = 0$. The left hand side is $(a-b)^2$, so we are assuming $(a-b)^2 = 0$. Thus, $a - b = 0$ so $a = b$. That is, $R$ is a square, as desired.

(b) Recall that if a number $p$ is prime, then for all divisors $d$ of $p$, either $d = p$ or $d = 1$. Prove that 15 is not prime, by using the contrapositive of the previous statement.

The contrapositive is that if there exists a divisor $d$ of $p$ which is not 1 or $p$, then $p$ is not prime.

For $p = 15$, take $d = 3$. 3 is a divisor of 15 as $3 \cdot 5 = 15$. But $3 \neq 1$ and $3 \neq 15$. So 15 is not prime.

Recall the big new definition from Monday's class: a *field* is a set $F$ together with two binary operations, denoted $+$ and $\cdot$, which satisfy all of the following properties (axioms):

(F 0) For all $a, b \in F$,

$$a + b \in F \qquad \text{and} \qquad a \cdot b \in F.$$

($F$ is *closed under addition* and *closed under multiplication*.)

(F 1) For all $a, b \in F$,

$$a + b = b + a \qquad \text{and} \qquad a \cdot b = b \cdot a.$$

(Addition and multiplication are both *commutative*.)

(F 2) For all $a, b, c \in F$,

$$(a + b) + c = a + (b + c) \qquad \text{and} \qquad (a \cdot b) \cdot c = a \cdot (b \cdot c).$$

(Addition and multiplication are both *associative*.)

(F 3) There exists an element $0 \in F$ such that, for all $a \in F$,

$$0 + a = a.$$

(There is an *additive identity* element. Furthermore, we proved it is unique.)
Likewise, there exists an element $1 \in F$ such that $1 \neq 0$ and, for all $a \in F$,

$$1 \cdot a = a.$$

(There is a *multiplicative identity* element. Furthermore, we proved it is unique.)

(F 4) For all $a \in F$,

$$\text{there exists } b \in F \text{ such that } a + b = 0.$$

(Every element has an *additive inverse*. Furthermore, we proved they are unique.)
Likewise, for all $a \in F$, *if* $a \neq 0$, then

$$\text{there exists } b \in F \text{ such that } a \cdot b = 1.$$

(Every element other than 0 has a *multiplicative inverse*.)

(F 5) For all $a, b, c \in F$,

$$a \cdot (b + c) = a \cdot b + a \cdot c.$$

(The *distributive* property, or "multiplication distributes over addition".)

You will need the above definition for all of the questions on this worksheet.

1. Fill in the conclusion for the following theorem, then prove the theorem.

   **Theorem.** *Let $F$ be a field. For all $a \in F$, $a \cdot 0 =$* $\underline{\ 0\ }$.

   (Note: In this case, the conclusion should be obvious, but the proof is actually pretty tricky! Try it on your own for a while, but if you're totally stuck, you can get a hint from your TA or an LA.)

   Let $a \in F$. $0+0=0$ by F3. Multiplying by $a$ yields $a\cdot(0+0)=a\cdot 0$. By F5, $a\cdot(0+0)=a\cdot 0+a\cdot 0$, so $a0+a0=a\cdot 0$. By F4, there is some $b \in F$ so that $a\cdot 0+b=0$. Then

   $$\begin{aligned} 0 &= a\cdot 0+b \\ &= (a\cdot 0+a\cdot 0)+b && \text{(F2)} \\ &= a\cdot 0+(a\cdot 0+b) \\ &= a\cdot 0+0 && \text{(F1)} \\ &= 0+a\cdot 0 \\ &= a\cdot 0 && \text{(F3)} \end{aligned}$$

   So $0=a\cdot 0$.

2. Is the set of integers, $\mathbb{Z}$, with the standard addition and multiplication operations, a field?

   If so, prove it. If not, exactly which of the field axioms are not satisfied?

   No. F4 is the only axiom that fails.

   Indeed, consider $2 \in \mathbb{Z}$. There is no $b \in \mathbb{Z}$ so that $2\cdot b=1$, as $\frac{1}{2} \notin \mathbb{Z}$.

3. Consider the following subset of the rational numbers:

$$X = \left\{ \frac{x}{y} \in \mathbb{Q} \;\middle|\; x \text{ and } y \text{ are odd} \right\}$$

Is this set $X$, with the standard addition and multiplication operations for rational numbers, a field?

If so, prove it. If not, exactly which of the field axioms are not satisfied?

F0 fails, as $\frac{1}{3} + \frac{1}{3} = \frac{2}{3} \notin X$

F3 fails, as $0 = \frac{0}{1} \notin X$

4. What other examples of fields can you come up with, other than $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$, and the other field(s) mentioned on this worksheet?

Some suggestions:

- Consider sets of matrices.
- Consider sets of certain types of functions.
- Consider subsets of the rational numbers like in problems 2 and 3.
- Consider sets containing all rational numbers, but not all irrational or complex numbers (so not all of $\mathbb{R}$ or $\mathbb{C}$).
- Consider other finite sets like $\mathbb{F}_2$.

- For a field $F$, let $F(t) = \left\{ \frac{f(t)}{g(t)} \;\middle|\; f, g \in P(F),\ g \neq 0 \right\}$. Then $F(t)$ is a field via $\frac{f_1(t)}{g_1(t)} + \frac{f_2(t)}{g_2(t)} = \frac{f_1(t)g_2(t) + g_1(t)f_2(t)}{g_1(t)g_2(t)}$ and $\frac{f_1(t)}{g_1(t)} \cdot \frac{f_2(t)}{g_2(t)} = \frac{f_1(t)f_2(t)}{g_1(t)g_2(t)}$

  So e.g. $\mathbb{R}(t)$ are the usual rational functions.

- $\mathbb{F}_4 = \{0, 1, a, b\}$ via

| + | 0 | 1 | a | b |
|---|---|---|---|---|
| 0 | 0 | 1 | a | b |
| 1 | 1 | 0 | b | a |
| a | a | b | 0 | 1 |
| b | b | a | 1 | 0 |

| · | 0 | 1 | a | b |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | a | b |
| a | 0 | a | b | 1 |
| b | 0 | b | 1 | a |