Def. $\displaystyle\sum_{\text{patterns } P} \underbrace{(\operatorname{sgn} P)}_{(-1)^{\#\text{swaps}}} \underbrace{(\text{prod } P)}_{\prod_{i=1}^{n} a_{i,P(i)}}$

A $n \times n$ matrix

2x2. $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \underline{ad - bc}$

3x3. $\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, $\left[\begin{matrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{matrix}\right]$

$(i, P(i))$

(a_{11}, a_{22}) $\rightarrow$ $aei + bfg + cdh - bdi - afh - ceg$

4x4 $\det\begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix} = ??$ $\underline{24}$ terms

4 multilinic

5x5 $\rightarrow$ 120 terms

6x6 $\rightarrow$ 720

$n \times n \rightarrow n!$

$n(n-1)(n-2)\cdots 2\cdot 1$

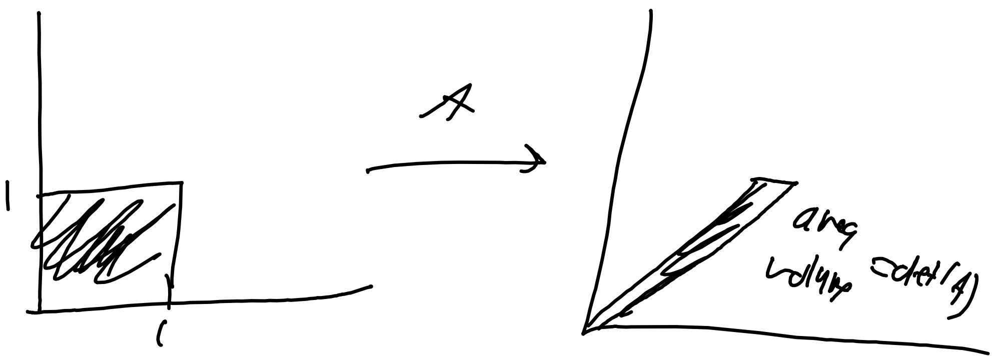
Properties

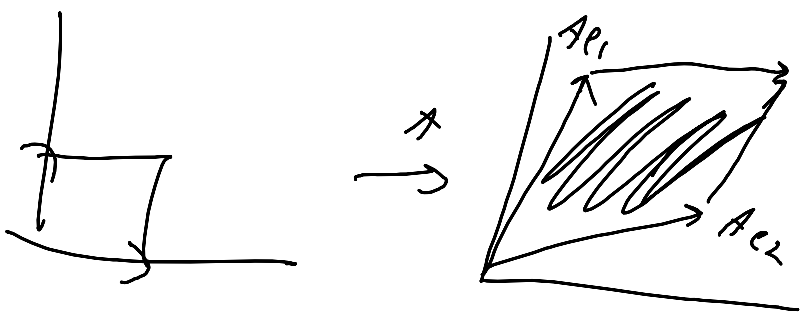
$\det(AB) = \det(A)\det(B)$

(warn: $\det(A+B) \neq \det(A) + \det(B)$)

$\det(I_2 - I_2) = 0$, $\det(I) + \det(-I)$
$= 1 + 1 = 2$
$0 \neq 2$

$\det(A) \neq 0 \iff A$ invertible





det(triangular matrices)

$$\det\begin{pmatrix} 1 & 4 & 5 \\ a & 2 & 6 \\ 0 & 0 & 3 \end{pmatrix} = 1\cdot 2\cdot 3 = 6$$

$$\det\begin{pmatrix} a_{11} & & * \\ & a_{22} & \\ & & \ddots & \\ 0 & & & a_{nn} \end{pmatrix} = \prod a_{ii} = a_{11}\, a_{22} \cdots a_{nn}$$

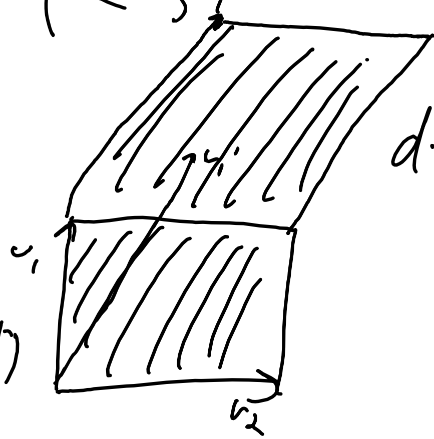
$$\begin{pmatrix} a_{11} & & 0 \\ & \ddots & \\ * & & a_{nn} \end{pmatrix} \qquad \det(A) = \det(A^t)$$

$$\det\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1$$

$$0\cdot 0 = 0 \neq$$

$$\frac{\det\begin{pmatrix} - & v_1 + v_1' & - \\ - & v_2 & - \end{pmatrix}}{= \det\begin{pmatrix} - & v_1 & - \\ - & v_2 & - \end{pmatrix} + \det\begin{pmatrix} - & v_1' & - \\ - & v_2 & - \end{pmatrix}}$$

$$\det\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} = \det\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} + \det\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}$$



$\det\begin{pmatrix} - & v_1' & - \\ - & v_2 & - \end{pmatrix}$

$\det\begin{pmatrix} - & v_1 & - \\ - & v_2 & - \end{pmatrix}$

$$\det\begin{pmatrix} - & v_1 + v_1' & - \\ - & v_2 & - \end{pmatrix}$$

$\boxed{\text{Computation}}$

Gauss–Jordan elimination

$$R_i \longmapsto R_i + \lambda R_j$$

$$A \xrightarrow[i \neq j]{} B$$

$$A = \begin{pmatrix} - & R_1 & - \\ & \vdots & \\ - & R_n & - \end{pmatrix} \longrightarrow \begin{pmatrix} - & R_1 & - \\ & \vdots & \\ - & R_i + \lambda R_j & - \\ & \vdots & \\ - & R_j & - \\ & \vdots & \\ - & R_n & - \end{pmatrix}$$

$$= \underbrace{\det \begin{pmatrix} - & R_1 & - \\ & \vdots & \\ - & R_i & - \\ & \vdots & \\ - & R_n & - \end{pmatrix}}_{\det(A)} + \underbrace{\det \begin{pmatrix} - & R_1 & - \\ & \vdots & \\ - & \lambda R_j & - \\ & \vdots & \\ - & R_j & - \\ & \vdots & \\ - & R_n & - \end{pmatrix}}_{\text{2 lin dep rows } = 0}$$

$$\underline{\det(A)} = \underline{\det(B)}$$

swapping rows

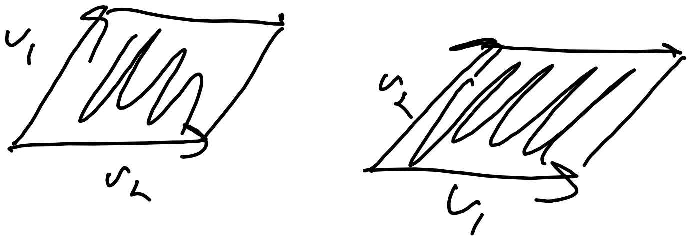
$$A \xrightarrow{R_i \leftrightarrow R_j} B$$

$$A = \begin{pmatrix} - & R_1 & - \\ & \vdots & \\ - & R_i & - \\ & \vdots & \\ - & R_j & - \\ & \vdots & \\ - & R_n & - \end{pmatrix} \longrightarrow B = \begin{pmatrix} - & R_1 & - \\ & \vdots & \\ - & R_j & - \\ & \vdots & \\ - & R_i & - \\ & \vdots & \\ - & R_n & - \end{pmatrix}$$

$$\underline{\det(B)} = -\underline{\det(A)}$$

<u>pattern</u> gets an additional swap

$$\underline{\operatorname{sgn} P} \rightsquigarrow -\operatorname{sgn} P$$

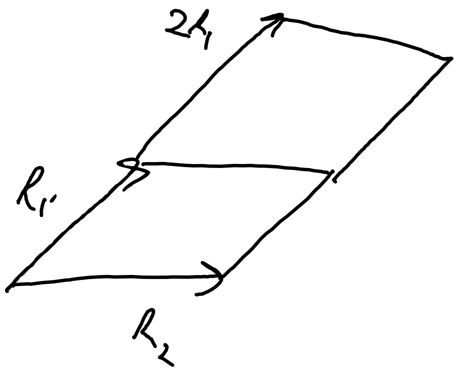


Scale

$$A \xrightarrow{R_i \mapsto \lambda R_i} B$$

$$A = \begin{pmatrix} - R_1 - \\ \vdots \\ - R_i - \\ \vdots \\ - R_n - \end{pmatrix} \Longrightarrow \begin{pmatrix} - R_1 - \\ \vdots \\ - \lambda R_i - \\ \vdots \\ - R_n - \end{pmatrix} B$$

$$\det(B) = \lambda \det(A)$$



$$\det \underbrace{\begin{pmatrix} 0 & 7 & 5 & 3 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & -1 \\ 1 & 1 & 1 & 2 \end{pmatrix}}_{A} \xleftrightarrow[(-1)]{R_1 \leftrightarrow R_2} \underbrace{\begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 7 & 5 & 3 \\ 1 & 1 & 2 & -1 \\ 1 & 1 & 1 & 2 \end{pmatrix}}_{B}$$

$$\xrightarrow[(1)]{R_3 \mapsto R_3 - R_1} \underbrace{\begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 7 & 5 & 3 \\ 0 & 0 & 0 & -2 \\ 1 & 1 & 1 & 2 \end{pmatrix}}_{C} \xrightarrow[(1)]{R_4 \mapsto R_4 - R_1} \underbrace{\begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 7 & 5 & 3 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & -1 & 1 \end{pmatrix}}_{D}$$

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 7 & 5 & 3 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & -1 & 1 \end{pmatrix} \xrightarrow[(-1)]{R_3 \leftrightarrow R_4} \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 7 & 5 & 3 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

D

$\det = 1 \cdot 7 \cdot -1 \cdot -2$

$= 14$

$$A \xrightarrow[(-1)]{R_1 \leftrightarrow R_2} B \xrightarrow[(1)]{R_3 \mapsto R_3 - R_1} C \xrightarrow[(1)]{R_4 \mapsto R_4 - R_1} D \xrightarrow[(-1)]{R_3 \leftrightarrow R_4} E$$

$-\det(B) = \det(A)$ $\quad \det(C) = \det(B)$ $\quad \det(D) = \det(C)$ $\quad \det(E) = -\det(D)$

$\det(A) = --\det(E) = \det(E) = 14$

$$\xrightarrow{R_2 \mapsto \frac{1}{7} R_2} \begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & -1 & \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

$\det(F) = 2$

$\det(E) = 7 \det(F)$

$\frac{1}{7} \det(E) = \det(F)$

$$\det\begin{pmatrix} - & R_1 & - \\ & \vdots & \\ - & 3R_i & - \\ & \vdots & \\ - & R_n & - \end{pmatrix} \quad R_i \mapsto 3R_i$$

$$= 3\det\begin{pmatrix} - & R_1 & - \\ & \vdots & \\ - & R_i & - \\ & \vdots & \\ - & R_n & - \end{pmatrix}$$

$$\det\begin{pmatrix} - & R_1 & - \\ - & R_2 & - \end{pmatrix}$$

$$\det\begin{pmatrix} - & 2R_1 & - \\ - & R_2 & - \end{pmatrix}$$

$$\| $$

$$2\det\begin{pmatrix} - & R_1 & - \\ - & R_2 & - \end{pmatrix}$$

double the area

$$\begin{pmatrix} 1 & & \\ & 2 & \\ & & 1 \end{pmatrix} \begin{pmatrix} - & R_1 & - \\ - & R_2 & - \\ - & R_3 & - \end{pmatrix}$$

$$\begin{pmatrix} - & R_1 & - \\ - & 2R_2 & - \\ - & R_3 & - \end{pmatrix}$$

$$\det \left(\begin{pmatrix} 1 & & \\ & 2 & \\ & & 1 \end{pmatrix} \begin{pmatrix} - & R_1 & - \\ - & R_2 & - \\ - & R_3 & - \end{pmatrix} \right)$$

$$= \det \begin{pmatrix} 1 & & \\ & 2 & \\ & & 1 \end{pmatrix} \det \begin{pmatrix} - & R_1 & - \\ - & R_2 & - \\ - & R_3 & - \end{pmatrix}$$

$$= 2 \det \begin{pmatrix} - & R_1 & - \\ - & R_2 & - \\ - & R_3 & - \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \to \det = 1$$

$$R_1 \longmapsto R_1 + R_2$$

Expansion by minors

Recursive

A $n \times n$ matrix

"expansion about the j^{th} column"

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \overbrace{\det(A_{ij})}^{\text{Smaller det}}$$

$$\begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & \ddots \\ - & + & & \\ + & & & \end{pmatrix}$$

A_{ij} is the $(n-1) \times (n-1)$ matrix which omits row i and column j

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$A_{11} = \begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix}$$

$$\det\begin{vmatrix} 1 & 0 & 1 & 2 \\ 4 & 1 & 3 & 0 \\ 4 & 2 & 2 & 0 \\ 5 & 0 & 0 & 3 \end{vmatrix} = -2\cdot -20 + 3\cdot 5 = 40 + 15 = 55$$

Moral: look for zeros

expand about col. 4

$$-2\cdot\underbrace{\det\begin{vmatrix} 4 & 1 & 3 \\ 4 & 2 & 2 \\ 5 & 0 & 0 \end{vmatrix}}_{-20} + \cancel{0\cdot\det(\dots)} \to 0 \quad - \cancel{0\cdot\det(\dots)} \to 0$$

$$+3\,\underbrace{\det\begin{vmatrix} 1 & 0 & 1 \\ 4 & 1 & 3 \\ 4 & 2 & 2 \end{vmatrix}}_{} = 5$$

expand about row 3

$$5\cdot\det\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix} - \cancel{0\cdot(\dots)} \to 0 \quad + \cancel{0\,(\dots)} \to 0$$

$$= 5\,(2-6) = -20$$

$$\det\begin{pmatrix} 1 & 0 & 1 \\ 9 & 1 & 3 \\ 9 & 2 & 2 \end{pmatrix}$$

expand about first row

$$1\cdot\det\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix} - \cancel{0\det\begin{pmatrix} 9 & 3 \\ 9 & 2 \end{pmatrix}} \quad 36$$

$$+1\cdot\det\begin{pmatrix} 9 & 1 \\ 9 & 2 \end{pmatrix}$$

$$(2-6) + (18-9)$$

$$= -4 + 9$$

$$= 5$$

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_0 & a_1 & \cdots & t \\ \vdots & \vdots & & \vdots \\ a_0^n & a_1^n & \cdots & t^n \end{pmatrix}$$

expand about last row

$1 \cdot (\text{minor})$

$- t (\text{minor})$

$+ t^2 (\text{minor})$

$\vdots$

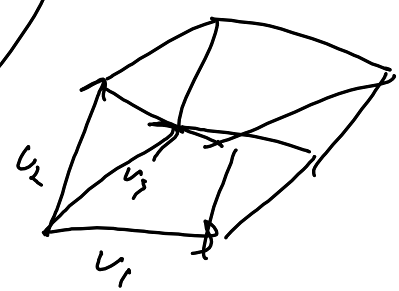
$+ t^n (\text{minor})$

deg n polynomial

$\vec{v}$, $\vec{w}$

$$\|\vec{v} \times \vec{w}\| = \underbrace{\|\vec{v}\| \, \|\vec{w}\| \sin(\theta)}$$

$$\det\begin{pmatrix} — \vec{v}_1 — \\ — \vec{v}_2 — \\ — \vec{v}_3 — \end{pmatrix} = \underbrace{\underbrace{(\vec{v}_1 \times \vec{v}_2)}_{\text{vecto}} \cdot \underbrace{\vec{v}_3}_{\text{vectr}}}_{\text{Scalar}}$$

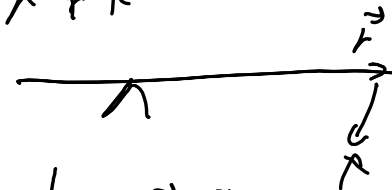


$$\det \begin{vmatrix} - & v_1 & - \\ - & v_2 & - \\ - & v_1 \times v_2 & - \end{vmatrix}$$

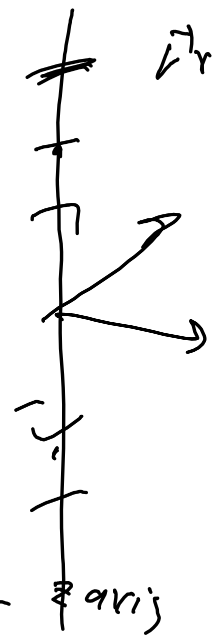
$= (v_1 \times v_2) \cdot (v_1 \times v_2)$

$= \|v_1 \times v_2\|^2$

$\vec{r} \times \vec{F}$

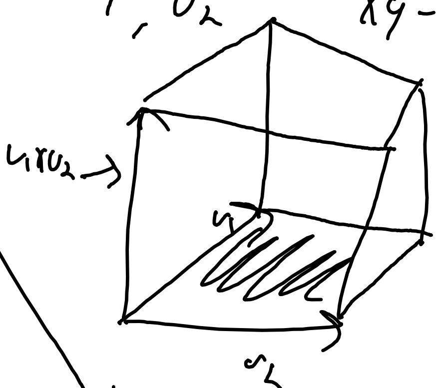


$\vec{r} \times \vec{F} = \vec{\tau}$



$v_1 \times v_2$ z axis

v_1, v_2 xy-plan



$$\|v_1 \times v_2\|^2 = \underbrace{\text{area}(v_1, v_2 \text{ parallelogram})}_{\|v_1 \times v_2\|} \cdot \|v_1 \times v_2\|$$

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \dots & v_n \\ | & | & & | \end{pmatrix}$$

$A = Q R$

Q: orthogonal

R: upper Δ

$R_{ii} = \|v_i^\perp\|$ $\quad R_{11} = \|v_1\|$

$\det(A) = \det(Q) \det(R)$

$\det(R) = \|v_1\| \prod_{i=1} \|v_i^\perp\|$

$-1 \leftarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$1 \leftarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Q orthog. $\quad Q^T = Q^{-1}$

$\det(Q^T) = \det(Q)$

$\quad \| $

$\det(Q^{-1})$

$\quad \|$

$1/\det(Q)$

$\frac{1}{\det(Q)} = \det(Q) \Rightarrow \det(Q)^2 = 1$

$\det(Q) = \pm 1$

$$|\det(A)| = \det(L)$$

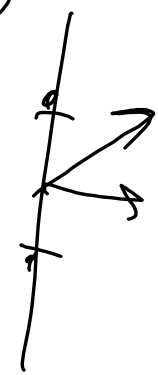
$$= \|v_1\| \prod \|v_i\|^{\perp}$$

orthogonal matrix

||

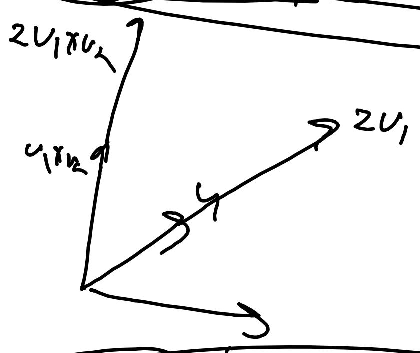
orthonormal matrix

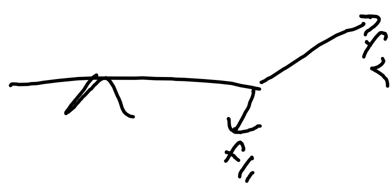
$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$



$\vec{v}_1 \times \vec{v}_2$

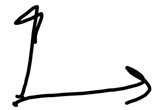
$$(c\vec{v}_1) \times \vec{v}_2 = c\,(\vec{v}_1 \times \vec{v}_2)$$

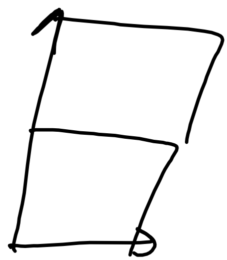




$$(\vec{v}_1 + \vec{v}_2) \times \vec{v}_3 = \vec{v}_1 \times \vec{v}_3 + \vec{v}_2 \times \vec{v}_3$$

$$\vec{e_1} \times \vec{e_2} = \vec{e_3}$$





$$\|(2e_1) \times e_2\| = 2$$

$A \quad 1$

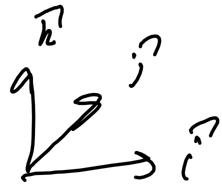
$-B \quad A^T$

$$\det\begin{pmatrix} \hat{i} & \hat{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} \begin{matrix} \\ v \\ w \end{matrix} = \vec{v} \times \vec{w}$$

$\hat{i} = \vec{e}_1$

$\hat{j} = \vec{e}_2$

$\vec{k} = \vec{e}_3$



$$(cv_1) \times v_2 = c(v_1 \times v_2)$$

$$v_1 \times v_2 = -v_2 \times v_1$$

n- vectors $\longrightarrow$ real #