The Intersection Product

on

Chow Groups

§1. The intersection product

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§1. The intersection product

Thm. Let X be a smooth variety.

There is a unique product structure on $CH^\bullet(X)$ making it into a commutative graded ring so that if $A, B \subseteq X$ are subvarieties intersecting generically transversely then

$$[A][B] = [A \cap B]$$

This extends to a functor

$$SmVar_k^{op} \xrightarrow{CH^\bullet} CGr(Ring)$$

Furthermore, for any algebraic scheme $Y_{/k}$ and morphism $Y \xrightarrow{f} X$, there is a cap product pairing

$$CH^p(X) \otimes CH_q(Y) \longrightarrow CH_{q-p}(Y)$$

making $CH_\bullet(Y)$ a $CH^\bullet(X)$-module. For f proper, f_* is a morphism of $CH^\bullet(X)$-modules, i.e. $f_*(f^*x \cdot y) = x \cdot f_*(y)$.

Rmk. Compare this to the cohomology ring $H^*(X(\mathbb{C});\mathbb{Z})$ of a smooth complex variety in its analytic topology.

Indeed, there is a cycle class map $CH^*(X) \to H^*(X(\mathbb{C});\mathbb{Z})$ taking a cycle class to its Poincaré dual.

An exterior product is easy to define.

Let X be smooth. We have

$$CH_p(X) \otimes CH_q(X) \longrightarrow CH_{p+q}(X \times X)$$
$$\alpha \otimes \beta \longmapsto \alpha \times \beta$$

Let $\Delta : X \hookrightarrow X \times X$ be the diagonal.

If we can form

$$CH_k(X \times X) \xrightarrow{\Delta^*} CH_{k-\dim(X)}(X)$$

then we will get a pairing

$$CH^p(X) \otimes CH^q(X) \longrightarrow CH^{p+q}(X)$$

Observation 1. X smooth implies that

$$\Delta : X \hookrightarrow X \times X$$

is a regular embedding, and hence has a normal bundle.

Observation 2. Let X be an algebraic scheme and $\pi : E \longrightarrow X$ a vector bundle on X of rank r.

Then flat pullback

$$CH_k(X) \xrightarrow{\pi^*} CH_{k+r}(E)$$

is an isomorphism.

Its inverse is "intersection with the 0-section", which is thus defined even in a singular setting!

Goal. Use these to construct "Gysin pullbacks" on Chow groups for any regular embedding.

Def. Let $Y \subseteq X$ be a closed subscheme.

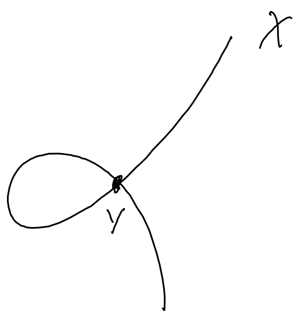
Let I be the ideal sheaf of $Y \subseteq X$.

The normal cone of Y in X is

$$C_{Y/X} = \underline{\mathrm{Spec}}_Y \left(\bigoplus_{n \geq 0} I^n / I^{n+1} \right)$$

Rmk. The exceptional divisor in $\mathrm{Bl}_Y X$ is $\mathbb{P}(C_{Y/X})$.

e.g. $Y = \{(0,0)\} \subseteq X = \{ y^2 = x^3 + x^2 \}$ in $\mathbb{A}^2$

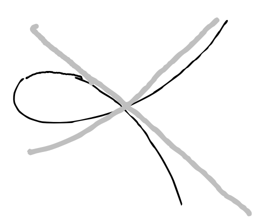


$$\bigoplus_{n \geq 0} \frac{(\bar{x}, \bar{y})^n}{(\bar{x}, \bar{y})^{n+1}} = k \oplus \frac{(\bar{x}, \bar{y})}{(\bar{x}, \bar{y})^2} \oplus \frac{(\bar{x}, \bar{y})^2}{(\bar{x}, \bar{y})^3} \oplus \cdots$$

This is generated by $\bar{x}, \bar{y}$ as a k-algebra.

$\bar{y}^2 = \bar{x}^3 + \bar{x}^2 \equiv \bar{x}^2 \mod (\bar{x}, \bar{y})^3$

So we get $k[x,y]/(x^2 - y^2)$



Thm (Deformation to the normal cone).

Let $Y \subseteq X$ be a closed subscheme.

There is a flat family of inclusions $Y \subseteq X_t$.

so that, $Y \subseteq X_0$ is $Y \subseteq X$

, $Y \subseteq X_\infty$ is $Y \subseteq C_{Y/X}$

Construction. $M = Bl_{Y \times \{\infty\}} X \times \mathbb{P}^1$, which has exceptional divisor

$\mathbb{P}(C_{Y \times \{\infty\} / X \times \mathbb{P}^1}) = \mathbb{P}(C_{Y/X} \oplus 1)$

Let $\tilde{X} = Bl_Y X \subseteq M$.

Let $M^\circ = M - \tilde{X}$. This is our flat family

e.g. $Y = \{(0,0)\}$, $X_t = \{y^2 = x^2 + tx^3\}$

Intersections ought to vary continuously in families, so we can often reduce to normal cones.

Now we construct the Gysin pullback of a regular embedding.

Let $Y \xhookrightarrow{i} X$ be a regular embedding

Let $f: V \longrightarrow X$.

$$\begin{tikzcd}
f^{-1}(X) \arrow[r, hook, "j"] \arrow[d, "g"'] & V \arrow[d, "f"] \\
Y \arrow[r, hook, "i"'] & X
\end{tikzcd}$$

Let $N = g^* N_{Y/X} \xrightarrow{\pi} f^{-1}(X)$

$C = C_{f^{-1}(X)/V}$

Let I be the ideal sheaf on $Y \subseteq X$, J the ideal sheaf of $f^{-1}(X) \subseteq V$.

we have $\bigoplus_{n \geq 0} f^*(I^n/I^{n+1}) \twoheadrightarrow \bigoplus_{n \geq 0} J^n/J^{n+1}$

so $C \subseteq N$.

Def: $X \cdot V = (\pi^*)^{-1}([C])$

This extends to $X \cdot \alpha \in CH(f^{-1}(X))$ for any $\alpha \in V$.

e.g. Suppose Y, Z are subvarieties of X intersecting transversely, and X, Y, Z are smooth.

$$\begin{tikzcd}
Y \cap Z \arrow[r] \arrow[d, hook, "g"'] & Y \times Z \arrow[d] \\
X \arrow[r, hook] & X \times X
\end{tikzcd}$$

i.e. $\forall\, p \in Y \cap Z$,

$$\dim T_p Y + \dim T_p Z = \dim T_p X$$

for any intersection point $p \in Y \cap Z$

Then $Y \cap Z \hookrightarrow Y \times Z$ is a regular embedding, so

$C = C_{Y\cap Z / Y \times Z} = N_{Y \cap Z / Y \times Z}$

$$\begin{aligned}
\dim(X) &= rk(N_{X/X\times X}) \\
&= rk(g^* N_{X/X\times X}) \\
&= rk(C)
\end{aligned}$$

so the inclusion $C \subseteq g^* N_{X/X\times X}$ is an equality.

Then
$$\begin{aligned}
X \cdot (Y \times Z) &= (\pi^*)^{-1}([C]) \\
&= (\pi^*)^{-1}(g^* N_{X/X\times X}) \\
&= (\pi^*)^{-1} \pi^* [Y \cap Z] \\
&= [Y \cap Z]
\end{aligned}$$

as we'd surely expect!

We now extend to Y and Z intersecting only <u>generically</u> transversely, i.e. at generic pts of components of the intersection.

Fact. The Gysin pullback is compatible w/ flat pullback.

That is, given

$$\begin{tikzcd} W' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & V' \arrow[d, "f' \text{ flat}"] \\ W \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & V \arrow[d, "f"] \\ Y \arrow[r, hook, "\text{regular}"'] & X \end{tikzcd}$$

Then $Y.f'^*\alpha = f'^* Y.\alpha$, for $\alpha \in CH(V)$

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ Y' \arrow[r, hook] \arrow[d] & X' \arrow[d] \\ Y \arrow[r, hook] & X \end{tikzcd}$$

$\alpha \in CH(V)$ $\quad X.\alpha \quad Y'.\alpha$

Apply this for f' an open embedding to show that Gysin pullback is well behaved with respect to restricting to open subsets.

To extend from generic transversality to any two cycles, we have the moving lemma.

Thm (Moving lemma). X smooth quasiprojective variety. $\alpha, \beta \in CH(X)$. There exist generically transverse cycles $A, B \in Z(X)$ so that $\alpha = [A]$ and $\beta = [B]$.

That $\alpha\beta := [A \cap B]$ is well defined is much harder.

§2. Chern classes and projective bundles

$\boxed{\text{Rmk. Heretofore assume existence of the intersection product.}}$

"Intersection with the 0-section" is defined via formally introducing operators $c_i(E) \cap (-)$ on Chow groups.

For smooth varieties these yield actual classes, but critically Chern classes of vector bundles act on Chow groups of any algebraic scheme.

Def. Let L be a line bundle on X.

For s a rational section of L, let

$$c_1(L) = \mathrm{div}(s)$$

in $CH^1(X)$.

In the smooth setting, this yields an isomorphism

$$\mathrm{Pic}(X) \longrightarrow CH^1(X)$$

This respects pullbacks along $f: X \longrightarrow Y$

$$f^*(c_1(L)) = c_1(f^* L)$$

Def. Let E be a vector bundle on a scheme X.

We let $\mathbb{P}(E) = \underline{\mathrm{Proj}}_X\left(\mathrm{Sym}(E^\vee)\right)$

$$\begin{array}{c} \mathbb{P}(E) \\ \big\downarrow \pi \\ X \end{array}$$

The fiber over $x \in X$ is $\mathbb{P}(E_x)$, the lines in E_x.

Thus, $\pi^* E$ has a tautological sub linebundle

$$\mathcal{O}_{\mathbb{P}(E)}(-1) \subseteq \pi^* E$$

where the fiber over $(x, \ell) \in \mathbb{P}(E)$ is $\ell \subseteq E_x$.

Let $\mathcal{O}_{\mathbb{P}(E)}(1) = \mathcal{O}_{\mathbb{P}(E)}(-1)^\vee$.

Let $\xi = c_1(\mathcal{O}_{\mathbb{P}(E)}(1)) \in CH^1(\mathbb{P}(E))$

Thm. Say $\mathrm{rk}(E) = r$ and X is smooth. Then

$\pi^*: CH^\bullet(X) \to CH^\bullet(\mathbb{P}(E))$ is injective and

$$CH^\bullet(\mathbb{P}(E)) = CH^\bullet(X)[\xi]/(f(\xi))$$

for a monic polynomial $f(\xi)$ of degree r.

Rmks. • The group theoretic part holds for any algebraic scheme X,

$$CH^\cdot(\mathbb{P}(E)) = \bigoplus_{i=0}^{r-1} CH^\cdot(X)\,\xi^i$$

where ξ acts as an operator.

• $CH^\cdot(\mathbb{P}^n) = \mathbb{Z}[x]/(x^{n+1})$ with $x = c_1(\mathcal{O}(1))$

Indeed, Josh showed that any linear subspace generates $CH^i(\mathbb{P}^n)$ for all i. The pairing

$$CH^i(\mathbb{P}^n) \otimes CH^{n-i}(\mathbb{P}^n) \longrightarrow \mathbb{Z}$$

shows that these are not torsion.

• If E is trivial then $\mathbb{P}(E) = X \times \mathbb{P}^{r-1}$, so by the Künneth formula

$$CH^\cdot(\mathbb{P}(E)) = CH^\cdot(X)[\xi]/(\xi^r)$$

• In topology, if we have a fiber bundle

$$\begin{array}{ccc} F & \longrightarrow & E \\ & & \downarrow \\ & & X \end{array}$$

where $H^*(E;\mathbb{Q}) \longrightarrow H^*(F;\mathbb{Q})$ is onto, then as $H^*(X;\mathbb{Q})$ module we have $H^*(E;\mathbb{Q}) \cong H^*(F;\mathbb{Q}) \otimes H^*(X;\mathbb{Q})$.
(Leray-Hirsch)

The existence of $\mathcal{O}_{\mathbb{P}(E)}(1)$ on $\mathbb{P}(E)$, which restricts to $\mathcal{O}(1)$ on all fibers, shows a similar surjectivity in the algebro-geometric context.

Pf. of thm.

Lemma. Let $\alpha \in CH^{\cdot}(X)$. Then

$$\pi_*(\varphi^i\alpha) = \begin{cases} \alpha & i = r-1 \\ 0 & i < r-1 \end{cases}$$

Pf. Here, $\varphi^i\alpha$ means $\varphi^i\pi^*\alpha$. By the projection formula,

$$\pi_*(\varphi^i\alpha) = \pi_*(\varphi^i)\alpha$$

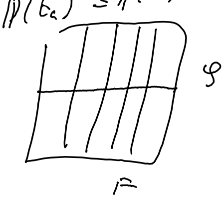
$$\pi_*(\varphi^i) \in CH_{\dim(X)+(r-1)+i}(X)$$

If $i < r-1$, $CH_{\dim(X)+r-1+i}(X) = 0$.

If $i = r-1$, $\pi_*(\varphi^{r-1}) \in CH^0(X) = \mathbb{Z}[X]$.

Thus, $\pi_*(\varphi^{r-1}) = m[X]$.

Let $a \in X$ be a closed point and $F = \mathbb{P}(E_a) \subseteq \mathbb{P}(E)$ be its fiber.



$$\begin{aligned} m = m[X]\cdot[a] &= \pi_*(\varphi^{r-1})[a] \\ &= \varphi^{r-1}[F] \\ &= \mathcal{O}(1)_{\mathbb{P}^{r-1}}^{r-1}[\mathbb{P}^{r-1}] \\ &= 1 \end{aligned}$$

□

Now, let

$$\begin{aligned} CH^\cdot(\mathbb{P}(E)) &\xrightarrow{\psi} \bigoplus_{i=0}^{r-1} CH^\cdot(X)\,\varphi^i \\ \beta &\longmapsto \sum_i \pi_*(\varphi^{r-i}\beta)\,\varphi^i \end{aligned}$$

$$\begin{aligned} \bigoplus_{i=0}^{r-1} CH^\cdot(X) &\longrightarrow CH^\cdot(\mathbb{P}(E)) \\ (\alpha_0, \dots, \alpha_{r-1}) &\longmapsto \sum \alpha_i \varphi^i \end{aligned}$$

We have $\psi\varphi = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ and is hence monic.

So for the addition isomorphism, we need only show $\sum CH^\cdot(X)\,\varphi^i = CH^\cdot(\mathbb{P}(E))$

This is a relative $(/X)$ version of the argument Josh gave that any linear subspace generates the Chow groups of $\mathbb{P}^n$.

Now, $\varphi^r \in \bigoplus_{i=0}^{r-1} CH^\cdot(X)\,\varphi^i$. Thus, we indeed have some monic polynomial $f \in CH^\cdot(X)[t]$ of deg r s.t. $f(\varphi) = 0$. Hence,

$$\begin{tikzcd} CH^\cdot(X)[t] \arrow[r, "t \mapsto \varphi"] \arrow[d] & CH^\cdot(\mathbb{P}(E)) \\ CH^\cdot(X)[\varphi]/(f(\varphi)) \arrow[ur, "\sim"'] & \end{tikzcd}$$

$\square$

Def. Let $f = \sum_{i=0}^{n} c_i(E)\varphi^i$.

We call $c_i(E)$ the i^{th} Chern class of E.

Rmk. We can explicitly define $c_i(E)$ via degeneracy loci of generating sections, but these needn't exist!

Thm. There is a unique assignment

$$E \mapsto c(E) = \sum c_i(E)$$

in $CH^*(X)$ so that

- $c(L) = 1 + c_1(L)$ as defined
- $s_0, \dots, s_{n-i}$ global sections of E degenerate on D of codim i, then $c_i(E) = [D]$
- $0 \to E \to F \to G \to 0 \Rightarrow c(G)c(E) = c(F)$
- $f: Y \to X$, then $f^*(c(E)) = c(f^*E))$.

Rmks. • These, as in topology, are pullbacks of (Schubert) classes in the Chow ring of Grassmannians.

Again, for a vector bundle E to determine a morphism to a Grassmannian we need sections!

• Consider $0 \longrightarrow \mathcal{O}_{\mathbb{P}(E)}(-1) \longrightarrow \pi^* E \longrightarrow Q \to 0$

on $\mathbb{P}(E)$. Then

$$c(Q)\, c(\mathcal{O}(-1)) = c(\pi^* E) = \pi^* c(E)$$

$c(\mathcal{O}(-1)) = 1 - \varphi$, and φ is nilpotent and hence we have $(1-\varphi)^{-1} = \sum \varphi^i$

Then $c(Q) = c(E) \sum \varphi^i$. Taking degree r parts, as Q has rank $r-1$, yields

$$0 = c_r(Q) = \sum_{i=0}^{r} c_i(E)\, \varphi^i$$

justifying a posteriori our definition of $c_i(E)$.

Cor. Let $\pi : E \longrightarrow X$ be a vector bundle.

Then flat pullback π^* is an isomorphism.

Pf. Surjectivity was via localization on trivializations, done last week.

We show injectivity.

Consider

$$\begin{tikzcd} \mathbb{P}(E \oplus \mathcal{O}) \arrow[d, "q"] \\ X \end{tikzcd}$$

Suppose $\pi^* \alpha = 0$. Let $E \overset{j}{\hookrightarrow} \mathbb{P}(E \oplus 1)$ as the complement of the hyperplane at ∞ $\mathbb{P}(E)$.

$$\begin{tikzcd} E \arrow[r, hook, "j"] \arrow[dr, "\pi"'] & \mathbb{P}(E \oplus 1) \arrow[d, "q"] & \mathbb{P}(E) \arrow[l, hook', "i"'] \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

Have $j^* q^* \alpha = 0$, so by localization, $q^* \alpha \in \operatorname{im}(i_*)$. Thus,

$$q^* \alpha = i_* \left(\sum p^* \alpha_i \, c_1(\mathcal{O}_E(1))^i \right)$$

for some $\boxed{\text{unique}}$ $\alpha_i \in CH_{k-(r-1)+i}(X)$

Lemma. $c_1(\mathcal{O}_{E\oplus 1}(1))\, q^* \alpha = i_* p^* \alpha$

Then $i_* \left(\sum p^* \alpha_i \, c_1(\mathcal{O}_E(1))^i \right)$

$\shortparallel$

$\sum q^* \alpha_i \, c_1(\mathcal{O}_{E\oplus 1}(1))^{i+1}$ (using projection and $i^* \mathcal{O}_{E\oplus 1}(1) = \mathcal{O}_E(1)$)

So $q^* \alpha = \sum q^* \alpha_i \, c_1(\mathcal{O}_{E\oplus 1}(1))^{i+1}$

in $CH^*(\mathbb{P}(E\oplus 1)) = \bigoplus_i q^* CH^*(X)\, c_1(\mathcal{O}_{E\oplus 1}(1))^i$

so both sides are in the φ^0 part of the direct sum. This can only happen if all the $\alpha_i = 0$, hence $\alpha = 0$. $\square$

§3. Blowups

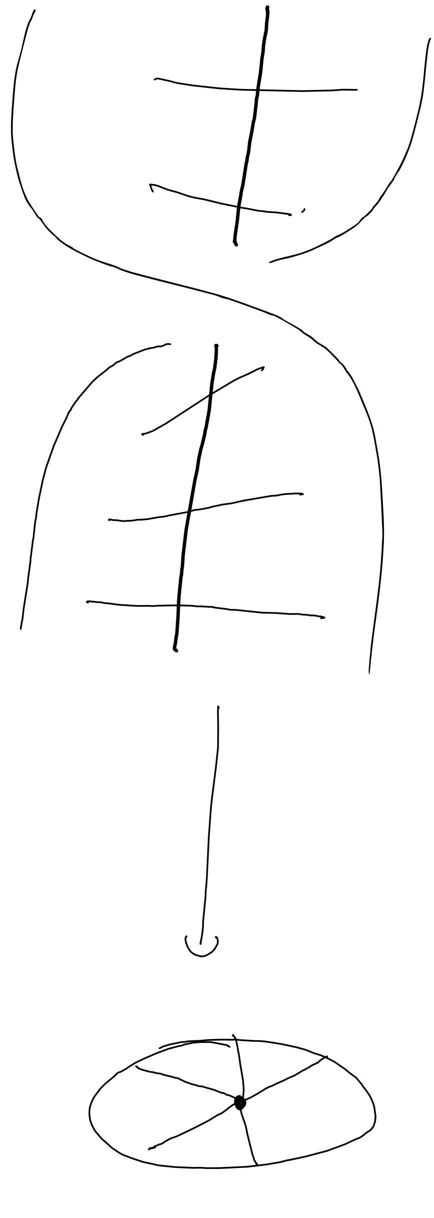
Def. Let $Z \hookrightarrow X$ be a closed subscheme and I its ideal sheaf. The blowup of $Z \hookrightarrow X$ is the universal pullback square

$$\begin{tikzcd} E_Z X \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Bl_Z X \arrow[d] \\ Z \arrow[r, hook] & X \end{tikzcd}$$

w/ $E_Z X \hookrightarrow Bl_Z X$ an effective Cartier divisor (locally defined by a single equation).

Explicitly,
$$Bl_Z X = \underline{\mathrm{Proj}}_X \left(\bigoplus_{i \geq 0} I^i \right)$$
$$E_Z X = \underline{\mathrm{Proj}}_Z \left(\bigoplus_{i \geq 0} I^i \otimes \mathcal{O}_Z \right)$$
$$= \underline{\mathrm{Proj}}_Z \left(\bigoplus_{i \geq 0} I^i / I^{i+1} \right)$$

e.g. $Bl_{(0,0)} \mathbb{A}^2 = \{ (x,y,\ell) \in \mathbb{A}^2 \times \mathbb{P}^1 \mid [x:y] = \ell \}$
$= \mathcal{O}_{\mathbb{P}^1}(-1)$



Suppose X, Z are smooth, $Z \overset{i}{\hookrightarrow} X$

Let $W = Bl_Z X$ and E be the exceptional divisor.

Then $E = \mathbb{P}(N_{Z/X})$ and $N_{E/W} = \mathcal{O}_E(-1)$.

Let $\varphi = c_1(\mathcal{O}_E(1))$

Let $\pi : W \to X$, $j : E \hookrightarrow W$.

Thm. $CH^*(W)$ is generated by $\pi^* (CH^*(X))$ and $j_* CH^*(E)$ (supported on E)

We have

- $\pi^*\alpha \cdot j_* \beta = j_*(\beta \cdot \pi|_E^* i^* \alpha)$
- $(j_* \beta)(j_* \gamma) = -j_*(\beta \gamma \varphi)$

Pf. Localization and proj. bundle formula.